

General proof of (maximum) entropy principle in Lovelock gravity

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We consider a static self-gravitating perfect fluid system in Lovelock gravity theory. For a spacial region on the hypersurface orthogonal to static Killing vector, by the Tolman's law of temperature, the assumption of a fixed total particle number inside the spacial region, and all of the variations (of relevant fields) in which the induced metric and its first derivatives are fixed on the boundary of the spacial region, then with the help of the gravitational and fluid equations of the theory, we can prove a theorem says that the total entropy of the fluid in this region takes an extremum value. A converse theorem can also be obtained following the reverse process of our proof. We also propose the definition of isolation quasilocally for the system and explain the physical meaning of the boundary conditions in the proof of the theorems.

I. INTRODUCTION

Black holes are fundamental objects in gravity theory which has been studied for a long time. Several breakthrough developments had been achieved about forty years ago. At the beginning of the 1970's, it is found that the laws of the mechanics of black holes are very similar to the usual four laws of thermodynamics [1]. Soon after, by studying the quantum effects of scalar field around a black hole, Hawking found that the black hole behaves like a blackbody with a temperature which is proportional to its surface gravity [2], and then the earlier proposal of the entropy of the black hole by Bekenstein could be confirmed to be one quarter of the horizon area [3]. Due to this celebrated work, the black hole mechanics is promoted to black hole thermodynamics. Since that time black hole thermodynamics has drawn a lot of attention, and has been widely studied during the past decades because people believe it might be a window in which one can catch sight of some important fundamental theories, such as the so-called quantum gravity theory.

Roughly speaking, there are two ways to approach the thermodynamics of black holes depending on the definitions of the thermodynamic quantities for associated spacetimes. Because of the equivalence principle in general relativity, some definitions of density for usual matter, such as energy density and entropy density are not valid for gravitational field. At most, we can define the energy of the gravitational field quasilocally. In the traditional construction of the thermodynamics of black holes, the thermodynamic quantities are identified to the global quantities defined at the infinities of the spacetimes, such as ADM mass, angular momentum, and charges of gauge fields. Fruitful results have been obtained along this way; for example, the thermodynamics of stationary black holes in general diffeomorphism invariant gravity theory has been established [4] (and references therein). However, in general, it is quite difficult to extract some useful local information of the spacetimes from this kind of thermodynamics of black holes. In 1993, Brown and York walked along another way and developed a new method to define the thermodynamic quantities quasilocally by using of a natural generalization of Hamilton-Jacobi analysis of action functional [5]. New notions on the definition of the horizons of black holes were also proposed soon after by some researchers [6, 7]. Since then, black hole thermodynamics can be studied quasilocally, and the gravitational equations can be obtained from these quasilocal thermodynamic quantities and associated thermodynamic relations with some additional assumptions. This quasilocal approach allows us to turn the logic around and study the gravitational equations from the laws of thermodynamics. Actually, Jacobson has shown that Einstein equations are the state equation which can be derived from the Clausius relation by using of local Rindler horizon and Unruh temperature [8, 9]. See also [10] and related works on some quasilocal horizons in dynamical spacetimes with spherical symmetry. These remarkable works inspire us to believe that the gravity and thermodynamics should have some deep and profound connection.

The thermodynamics of black holes heavily depends the quantum field theory in curved spacetimes. Quantum effects allow us to regard black holes as real thermodynamic systems. However, besides the black holes, there are also a lot of self-gravitating systems without horizons in general relativity and other possible gravity theories. Of course, the thermodynamics of these self-gravitating systems is very different from the thermodynamics of black holes. For example, the Hawking temperature does not exist in these systems. According to the work of Jacobson, we know that the gravitational equations can be deduced from the laws of thermodynamics. (This work is based on the thermodynamics of the local Rindler horizon, and some results from the quantum field in curved spacetimes have been used, such as the Unruh effect.) So a question naturally arises - whether it is possible to get the gravitational equations from the thermodynamics of the usual matter fields living in the curved spacetimes? Although

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there are no horizons and black holes in these cases, the gravitational equations and the laws of thermodynamics govern the same thing, i.e., the distribution of the matter fields (or equilibrium state of the matter fields) in stationary spacetimes. So some equivalent description among gravitational equations and thermodynamic laws might exist for these self-gravitating systems. For a spherical radiation system, Sorkin, Wald, and Zhang (SWZ) have shown that one can deduce the Tolman-Oppenheimer-Volkoff (TOV) equation from Hamiltonian constraint when the total entropy of radiation is in extremum [11]. Gao generalized SWZ's work to an arbitrary perfect fluid in static spherical spacetime and successfully got the TOV equation for this fluid [12]. Recently, a more general proof of the (maximum) entropy principle in the case of static spacetime without the spherical symmetry has been completed in [13, 14].

However, all of the above discussions are limited in Einstein gravity. One can ask whether the entropy principle is still valid or not in other gravity theories. Lovelock gravity theory is a natural generalization of Einstein gravity to higher dimensions. The action of this theory includes higher derivative terms with respect to metric, while the equations of motion still keep the derivatives up to second order [15]. Due to the development of supergravity and string theory, Lovelock gravity theory becomes more and more important. For example, people find that the higher order Lovelock terms appear in the higher order α' expansion of string amplitude [16–20]. Thus it is interesting to discuss the (maximum) entropy principle in this generalized gravity theory. The (maximum) entropy principle for the self-gravitating perfect fluid in an n -dimensional Lovelock gravity with the symmetry of an $(n - 2)$ -dimensional maximally symmetric space has been studied by the present authors in [21], where the generalized TOV equations have been derived from both gravity field equations and the (maximum) entropy principle of perfect fluid.

In this paper, we will present a proof of a theorem that the (maximum) entropy principle in the Lovelock gravity theory generally holds without considering the symmetry of an $(n - 2)$ -dimensional maximally symmetric space. The only symmetry we will consider is the static condition of the self-gravitating system. Assuming that the Tolman's law of the temperature holds for the perfect fluid in curved spacetimes, and imposing some boundary conditions in the variations of relevant fields, we show that the entropy of the fluid inside an $(n - 2)$ -dimensional spacelike surface [which is embedded in an $(n - 1)$ -dimensional hypersurface orthogonal to the static Killing vector of the spacetime] takes extremum value. Our discussion is focused on the system inside the $(n - 2)$ -dimensional spacelike surface of the static spacetimes, so it is not hard to extract some local information, i.e., a part of gravitational equations, from the law of thermodynamics of the system. This suggests that the converse theorem can be read out from our proof. Related work on the thermodynamics of self-gravitating system can be found in Ref. [22] in which global quantities, such as ADM mass, have been used to discuss the thermodynamic stability of the system.

We also study the physical meaning of boundary conditions. We found that the boundary conditions which seem nontransparent in the proof of the theorems finally turn out to be the isolation condition which is necessary of applicability of (maximum) entropy principle. Since the system we consider here is a quasilocal system, so the isolation here is quasilocally defined. In this sense, our proof is self-contained.

This paper is organized as follows: In Sec.II, for static spacetime, we study the equations of motion of the so-called Einstein-Gauss-Bonnet gravity which can be viewed as a special case of the Lovelock gravity up to second order. In Sec.III, we study the thermodynamics of the perfect fluid in the Einstein-Gauss-Bonnet theory and prove a theorem which relates the gravitational equations and the (maximum) entropy principle of the fluid. In Sec.IV, we generalize our proof to the general Lovelock gravity theory. In Sec.V, we check our previous work in which we studied the entropy principle in Lovelock gravity with an $(n - 2)$ -dimensional maximally symmetric space by using the present method. In Sec.VI, in contrast to the isolation condition of the usual thermodynamic system, we present the definition of an isolated system quasilocally. The last section is devoted to some conclusions and discussion.

II. THE EQUATIONS OF MOTION OF EINSTEIN-GAUSS-BONNET GRAVITY IN STATIC SPACETIMES

The Einstein-Gauss-Bonnet gravity is a typical example of the general Lovelock gravity. As a warm-up and an example, in this section, we consider the case of the Einstein-Gauss-Bonnet gravity in an n -dimensional spacetime (M, g_{ab}) . The action of this system can be written as

$$I = \frac{1}{2} \int \epsilon (\mathcal{R} + \alpha \mathcal{L}_{GB}) + I_{\text{matter}}, \quad (2.1)$$

where α is the so-called Gauss-Bonnet coupling constant, ϵ is the natural volume element associated with the metric g_{ab} , and \mathcal{L}_{GB} is the Gauss-Bonnet term which has a form

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}_{abcd}\mathcal{R}^{abcd}. \quad (2.2)$$

Here, we have used curly alphabet to denote the geometric quantities in the n -dimension. For instance, \mathcal{R}_{abcd} , \mathcal{R}_{ab} , and \mathcal{R} are the curvature tensor, Ricci tensor, and scalar curvature for the n -dimensional spacetime respectively. The symbol I_{matter} represents the action for the matter fields. The variation of the action with respect to the metric yields the gravitational equation

$$\mathcal{G}_{ab} + \alpha \mathcal{H}_{ab} = T_{ab}, \quad (2.3)$$

where \mathcal{G}_{ab} is the Einstein tensor of the spacetime (M, g_{ab}) and \mathcal{H}_{ab} is given by

$$\mathcal{H}_{ab} = 2\mathcal{R}_{acde}\mathcal{R}_b{}^{cde} - 4\mathcal{R}^{cd}\mathcal{R}_{acbd} - 4\mathcal{R}_{bc}\mathcal{R}_a{}^c + 2\mathcal{R}\mathcal{R}_{ab} - \frac{1}{2}\mathcal{L}_{GB} g_{ab}. \quad (2.4)$$

Since we have set $8\pi G = 1$, the right-hand side of Eq.(2.3) is simply the energy-momentum tensor T_{ab} for the matter fields.

The spacetime (M, g_{ab}) we are considering is assumed to be stationary. This suggests that we have a timelike Killing vector field K^a , i.e., K^a satisfies Killing equation $\nabla_{(a}K_{b)} = 0$, where ∇_a is the covariant derivative compatible with the metric g_{ab} (we use the notations and conventions in [23]). Following the notation by Geroch [24], in the region where $\lambda = K^a K_a \neq 0$, we can define a metric on the orbit space Σ of the Killing field

$$h_{ab} = g_{ab} - \lambda^{-1} K_a K_b. \quad (2.5)$$

If the Killing vector field is hypersurface orthogonal, i.e., Frobenius condition

$$K_{[a}\nabla_b K_{c]} = 0 \quad (2.6)$$

is satisfied, the orbit space Σ can be viewed as a hypersurface embedded in the spacetime. We always assume this condition is satisfied in following discussion. In other words, the spacetime is further assumed to be static. Considering the Frobenius condition (2.6), it is not hard to find

$$R_{abcd} = \mathcal{R}_{abcd} - 2\lambda^{-2} K_{[a}\nabla_{b]}\nabla_{[c}\lambda K_{d]} + \lambda^{-3} K_{[a}\nabla_{b]}\lambda\nabla_{[c}\lambda K_{d]}. \quad (2.7)$$

Here, R_{abcd} is the intrinsic curvature of the hypersurface (Σ, h_{ab}) . Based on the relation, we have the following decompositions

$$\mathcal{R} = R - \frac{1}{2}\lambda^{-1}h_{ab}(2D^a D^b \lambda - \lambda^{-1}D^a \lambda D^b \lambda), \quad (2.8)$$

and

$$\mathcal{L}_{GB} = L_{GB} + 2\lambda^{-1}G_{ab}(2D^a D^b \lambda - \lambda^{-1}D^a \lambda D^b \lambda), \quad (2.9)$$

where D_a is the covariant derivative operator which is compatible with the induced metric h_{ab} in Eq.(2.5), and R , L_{GB} , and G_{ab} are the scalar curvature, Gauss-Bonnet term, and, Einstein tensor for the hypersurface (Σ, h_{ab}) respectively. Furthermore, we have

$$\mathcal{G}_{ab} = G_{ab} - \frac{1}{2}\lambda^{-1}R K_a K_b + \frac{1}{2}h_{a[b}h_{d]c}(2\lambda^{-1}D^c D^d \lambda - \lambda^{-2}D^c \lambda D^d \lambda), \quad (2.10)$$

and

$$\begin{aligned} \mathcal{H}_{ab} = & H_{ab} - \frac{1}{2}\lambda^{-1}L_{GB}K_a K_b + \left[R_{acbd} - 2R_{a[b}h_{d]c} + 2R_{c[b}h_{d]a} \right. \\ & \left. + R h_{a[b}h_{d]c} \right] (2\lambda^{-1}D^c D^d \lambda - \lambda^{-2}D^c \lambda D^d \lambda), \end{aligned} \quad (2.11)$$

with

$$H_{ab} = 2R_{acde}R_b{}^{cde} - 4R^{cd}R_{acbd} - 4R_{bc}R_a{}^c + 2R R_{ab} - \frac{1}{2}L_{GB} h_{ab}. \quad (2.12)$$

From the above equations, it is easy to find following relations

$$\lambda^{-1}\mathcal{G}_{ab}K^a K^b = -\frac{1}{2}R, \quad (2.13)$$

and

$$\lambda^{-1}\mathcal{H}_{ab}K^a K^b = -\frac{1}{2}L_{GB}, \quad (2.14)$$

and

$$h_a{}^c h_b{}^d \mathcal{G}_{cd} = G_{ab} + \frac{1}{2}h_{a[b}h_{d]c}(2\lambda^{-1}D^c D^d \lambda - \lambda^{-2}D^c \lambda D^d \lambda), \quad (2.15)$$

and

$$h_a{}^c h_b{}^d \mathcal{H}_{cd} = H_{ab} + \left[R_{acbd} - 2R_{a[b}h_{c]d} + 2R_{c[b}h_{d]a} + R h_{a[b}h_{d]c} \right] (2\lambda^{-1}D^c D^d \lambda - \lambda^{-2}D^c \lambda D^d \lambda). \quad (2.16)$$

These relations are important in our proof of the entropy theorem in the next section.

III. SELF-GRAVITATING FLUID IN STATIC SPACETIME AND (MAXIMUM) ENTROPY PRINCIPLE

In this section, we are going to analyze the self-gravitating fluid in the Einstein-Gauss-Bonnet gravity. In the static spacetime (M, g_{ab}) , we assume that the Tolman's law holds, which says that the local temperature T of the fluid satisfies

$$T\sqrt{-\lambda} = T_0, \quad (3.1)$$

where T_0 is a constant which can be viewed as the local temperature of the fluid at some reference points with $\lambda = -1$. This relation is essential in the construction of an equilibrium state matter distribution in a curved spacetime, and it is popular satisfied in general stationary systems. Without loss of generality we shall take $T_0 = 1$.

Now, we can present a theorem which relates the (maximum) entropy principle of the fluid and the equations of motion in this static spacetime listed in the previous section.

Theorem 1 - Consider a self-gravitating perfect fluid in a static n -dimensional spacetime (M, g_{ab}) in Einstein-Gauss-Bonnet gravity and Σ is an $(n - 1)$ -dimensional hypersurface orthogonal to the static Killing vector. Let C be a region inside Σ with a boundary ∂C and h_{ab} be the induced metric on Σ . Assume that the temperature of the fluid obeys Tolman's law and equations of motion of both gravity and fluid are satisfied in C . Then the fluid is distributed such that its total entropy in C is an extremum for fixed total particle number in C and for all the variations in which h_{ab} and its first derivatives are fixed on ∂C .

Proof. - The integral curves of the static Killing vector field K^a can be viewed as some static observers in the spacetime, and the velocity vector field of such observers are given by

$$u^a = \frac{K^a}{\sqrt{-\lambda}}. \quad (3.2)$$

Obviously, u^a is just the unit norm of the hypersurface Σ . The acceleration vector associated with these observers has a form

$$A_a = \frac{\nabla_a \lambda}{2\lambda}. \quad (3.3)$$

For a general perfect fluid as discussed in [21], the energy-momentum tensor T_{ab} takes a form

$$T_{ab} = \rho u_a u_b + p h_{ab}, \quad (3.4)$$

where ρ and p are the energy density and pressure of the fluid respectively. In another word, u^a is also the velocity of the comoving observers of the fluid. The entropy density s is taken to be a function of the energy density ρ and particle number density n (do not confuse with the dimension of the spacetime), i.e., $s = s(\rho, n)$. The standard first law of thermodynamics in terms of these densities and Gibbs-Duhem relation are listed here as follows

$$d\rho = Tds + \mu dn, \quad (3.5)$$

$$s = \frac{1}{T}(\rho + p - \mu n), \quad (3.6)$$

where μ is the chemical potential conjugating to the particle number density n . It can be shown from conservation law $\nabla_a T^{ab} = 0$ and static conditions one gets

$$\nabla_a p = -(\rho + p)A_a = -(\rho + p)\frac{\nabla_a \lambda}{2\lambda}, \quad (3.7)$$

together with Eqs.(3.1), (3.5), and (3.6), we find

$$\frac{\nabla_a \mu}{\mu} = -\frac{\nabla_a \lambda}{2\lambda}, \quad (3.8)$$

which leads to

$$\mu\sqrt{-\lambda} = \text{constant}, \quad (3.9)$$

or

$$\frac{\mu}{T} = \text{constant}. \quad (3.10)$$

The total entropy S inside the region C on Σ is defined as the integral of the entropy density, i.e.,

$$S = \int_C \bar{\epsilon} s(\rho, n), \quad (3.11)$$

where $\bar{\epsilon}$ is the volume element of Σ associated with the induced metric h_{ab} , and invoking the local first law of thermodynamics (3.5), the variation of total entropy yields

$$\begin{aligned} \delta S &= \int_C \left[s \delta \bar{\epsilon} + \bar{\epsilon} \left(\frac{\partial s}{\partial \rho} \delta \rho + \frac{\partial s}{\partial n} \delta n \right) \right] \\ &= \int_C \left[s \delta \bar{\epsilon} + \bar{\epsilon} \left(\frac{1}{T} \delta \rho - \frac{\mu}{T} \delta n \right) \right]. \end{aligned} \quad (3.12)$$

Similarly, the total number of particle N is an integral

$$N = \int_C \bar{\epsilon} n. \quad (3.13)$$

So the fixed total particle number in C yields the following constraint

$$\int_C \bar{\epsilon} \delta n = - \int_C n \delta \bar{\epsilon}. \quad (3.14)$$

With this constraint, the variation of the total entropy becomes

$$\begin{aligned} \delta S &= \int_C \bar{\epsilon} \left(\frac{1}{2} \frac{\rho + p}{T} h^{ab} \delta h_{ab} + \frac{1}{T} \delta \rho \right) \\ &= \int_C \left[\bar{\epsilon} \frac{p}{2T} h^{ab} \delta h_{ab} + \frac{1}{T} \delta(\bar{\epsilon} \rho) \right], \end{aligned} \quad (3.15)$$

where we have used $\mu/T = \text{constant}$ and $\delta \bar{\epsilon} = (1/2) \bar{\epsilon} h^{ab} \delta h_{ab}$. The variations we perform here are only restricted on spacelike hypersurface Σ .

For the perfect fluid in an equilibrium state, the total entropy must take maximal value according to the maximum entropy principle. So our purpose is to proof the extremum condition $\delta S = 0$ from the gravitational equations which has been studied in sec.??.

From Eqs. (2.13) and (2.14), one can easily get the Hamiltonian constraint of the theory

$$\rho = \frac{1}{2} (R + \alpha L_{GB}), \quad (3.16)$$

while Eqs.(2.15) and (2.16) lead to evolution equations

$$\begin{aligned} p h^{ab} &= G^{ab} + \alpha H^{ab} + \frac{1}{2} h^{a[b} h^{d]c} (2\lambda^{-1} D_c D_d \lambda - \lambda^{-2} D_c \lambda D_d \lambda) + \alpha [R^{acbd} - 2R^{a[b} h^{c]d}] \\ &\quad + 2R^{c[b} h^{d]a} + R h^{a[b} h^{d]c}] (2\lambda^{-1} D_c D_d \lambda - \lambda^{-2} D_c \lambda D_d \lambda). \end{aligned} \quad (3.17)$$

We can calculate the second term on the right-hand side of Eq.(3.15) by using the Hamiltonian constraint (3.16). Actually, we have

$$\int_C \frac{1}{T} \delta(\bar{\epsilon} \rho) = \frac{1}{2} \int_C \frac{1}{T} \delta[\bar{\epsilon} (R + \alpha L_{GB})] = \frac{1}{2} \int_C \bar{\epsilon} \frac{1}{T} [- (G^{ab} + \alpha H^{ab}) \delta h_{ab} + \mathcal{B}_1 + \alpha \mathcal{B}_2], \quad (3.18)$$

where \mathcal{B}_1 and \mathcal{B}_2 are total derivative terms which come from the variation of the Einstein-Hilbert term and Gauss-Bonnet term in the right-hand side of Eq.(3.16) respectively. Explicitly, they are given by

$$\mathcal{B}_1 = D^a D^b \delta h_{ab} - D^a (h^{bc} D_a \delta h_{bc}), \quad (3.19)$$

and

$$\begin{aligned} \mathcal{B}_2 &= 2D^a (R D^b \delta h_{ab}) - 2D^b (D^a R \delta h_{ab}) \\ &\quad - 2D^a (R h^{bc} D_a \delta h_{bc}) + 2D_a (D^a R h^{bc} \delta h_{bc}) \\ &\quad - 8D_b (R^{ac} h^{bd} D_a \delta h_{cd}) + 8D_a (D_b R^{ac} h^{bd} \delta h_{cd}) \\ &\quad + 4D_b (R^{ac} h^{bd} D_d \delta h_{ac}) - 4D_d (D_b R^{ac} h^{bd} \delta h_{ad}) \\ &\quad + 4D_a (R^{ac} h^{bd} D_c \delta h_{bd}) - 4D_c (D_a R^{ac} h^{bd} \delta h_{bd}) \\ &\quad + 2D_b (R^{abcd} D_c \delta h_{ad}) - 2D_c (D_b R^{abcd} \delta h_{ad}) \\ &\quad - 2D_b (R^{abcd} D_d \delta h_{ac}) + 2D_d (D_b R^{abcd} \delta h_{ac}). \end{aligned} \quad (3.20)$$

By using integration by parts, and dropping the surface terms with fixed h_{ab} and its first derivative, one can rewrite the last two terms of Eq.(3.18) as follows

$$\begin{aligned}
\frac{1}{2} \int_C \bar{\epsilon} \frac{\mathcal{B}_1}{T} &= \frac{1}{2} \int_C \bar{\epsilon} \frac{1}{T} \left[D^a D^b \delta h_{ab} - D^a (h^{bc} D_a \delta h_{bc}) \right] \\
&= \frac{1}{2} \int_C \bar{\epsilon} \left[D^a D^b \left(\frac{1}{T} \right) - D^c D_c \left(\frac{1}{T} \right) h^{ab} \right] \delta h_{ab} \\
&= - \int_C \bar{\epsilon} h^{a[b} h^{d]c} D_c D_d \left(\frac{1}{T} \right) \delta h_{ab}, \tag{3.21}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2} \int_C \bar{\epsilon} \frac{\mathcal{B}_2}{T} &= \frac{1}{2} \int_C \bar{\epsilon} \left[2D^a D^b \left(\frac{1}{T} \right) R \right. \\
&\quad - 2D^c D_c \left(\frac{1}{T} \right) R h^{ab} - 8D_c D_d \left(\frac{1}{T} \right) R^{ca} h^{db} \\
&\quad + 4D_c D_d \left(\frac{1}{T} \right) R^{ab} h^{cd} + 4D_c D_d \left(\frac{1}{T} \right) R^{cd} h^{ab} \\
&\quad + 2D_c D_d \left(\frac{1}{T} \right) R^{adcb} - 2D_c D_d \left(\frac{1}{T} \right) R^{acbd} \\
&\quad - 4D_c \left(\frac{1}{T} \right) h^{ab} (D^c R - 2D_d R^{cd}) \\
&\quad + 4D^a \left(\frac{1}{T} \right) (D^b R - 2D_c R^{cb}) \\
&\quad \left. - 8D_c \left(\frac{1}{T} \right) (D^a R^{cb} - D^c R^{ab} + D_d R^{acbd}) \right] \delta h_{ab} \\
&= - \int_C \bar{\epsilon} \left[2R h^{a[b} h^{d]c} D_c D_d \left(\frac{1}{T} \right) - 4(R^{a[b} h^{c]d} \right. \\
&\quad \left. - R^{c[b} h^{d]a}) D_c D_d \left(\frac{1}{T} \right) + 2R^{acbd} D_c D_d \left(\frac{1}{T} \right) \right] \delta h_{ab}, \tag{3.22}
\end{aligned}$$

where all of the terms with the derivatives of Riemann tensor are eliminated by using Bianchi identity.

Remembering that T_0 has been set to be a unit, so we have following relation

$$2\lambda^{-1} D_c D_d \lambda - \lambda^{-2} D_c \lambda D_d \lambda = 4T D_c D_d \left(\frac{1}{T} \right). \tag{3.23}$$

Thus the evolution equations become

$$p h^{ab} = G^{ab} + \alpha H^{ab} + 2h^{a[b} h^{d]c} T D_c D_d \left(\frac{1}{T} \right) + 4\alpha \left[R^{acbd} - 2R^{a[b} h^{c]d} + 2R^{c[b} h^{d]a} + R h^{a[b} h^{d]c} \right] T D_c D_d \left(\frac{1}{T} \right). \tag{3.24}$$

Combining Eqs. (3.18), (3.21), (3.22), and (3.24), we find that the variation of total entropy (3.15) is exactly vanishing, i.e., we have

$$\delta S = 0. \tag{3.25}$$

So far we have completed our proof of Theorem 1.

IV. MAXIMUM ENTROPY PRINCIPLE IN LOVELOCK GRAVITY

In the previous sections, we have proved our Theorem 1 in the context of Einstein-Gauss-Bonnet gravity. However, the theorem can also be generalized to a more general Lovelock gravity. The Lovelock action in n -dimensional spacetime is given by

$$I = \frac{1}{2} \int \epsilon \sum_{i=0}^{[n/2]} \alpha_{(i)} \mathcal{L}_{(i)} + I_{\text{matter}}, \tag{4.1}$$

where ϵ is still the volume element of the static spacetime (M, g_{ab}) , and I_{matter} also represents the action of matter fields. The coefficients $\alpha_{(i)}$ are constants and $\mathcal{L}_{(i)}$ is defined as

$$\mathcal{L}_{(i)} = \frac{1}{2^i} \delta_{c_1 \dots c_i d_1 \dots d_i}^{a_1 \dots a_i b_1 \dots b_i} \mathcal{R}_{a_1 b_1}{}^{c_1 d_1} \dots \mathcal{R}_{a_i b_i}{}^{c_i d_i}. \tag{4.2}$$

Here, $\delta_{c_1 \dots c_i d_1 \dots d_i}^{a_1 \dots a_i b_1 \dots b_i} = (2i)! \delta_{c_1}^{a_1} \dots \delta_{c_i}^{a_i} \delta_{d_1}^{b_1} \dots \delta_{d_i}^{b_i}$ is generalized Kronecker delta symbol. Considering the variation with respect to g_{ab} , one gets the equations of motion which havw a form

$$\mathcal{G}_{ab} = \sum_{i=0}^{[n/2]} \alpha_{(i)} \mathcal{G}_{ab}^{(i)} = T_{ab}, \quad (4.3)$$

where

$$\mathcal{G}_{ab}^{(i)a} = -\frac{1}{2^{i+1}} \delta_{bc_1 \dots c_i d_1 \dots d_i}^{aa_1 \dots a_i b_1 \dots b_i} \mathcal{R}_{a_1 b_1}{}^{c_1 d_1} \dots \mathcal{R}_{a_i b_i}{}^{c_i d_i}. \quad (4.4)$$

For the static spacetime (M, g_{ab}) , by using Eq.(2.7), after lengthy and tedious calculation, we find

$$\mathcal{L}_{(i)} = L_{(i)} + i G_{ab}^{(i-1)} (2\lambda^{-1} D^a D^b \lambda - \lambda^{-2} D^a \lambda D^b \lambda), \quad (4.5)$$

and

$$\mathcal{G}_{ab}^{(i)} = G_{ab}^{(i)} - \frac{1}{2} \lambda^{-1} L_{(i)} K_a K_b + E_{acbd}^{(i)} (2\lambda^{-1} D^c D^d \lambda - \lambda^{-2} D^c \lambda D^d \lambda), \quad (4.6)$$

where $L_{(i)}$ is the i th Lovelock Lagrangian on the hypersurface (Σ, h_{ab}) and has an expression

$$L_{(i)} = \frac{1}{2^i} \delta_{c_1 \dots c_i d_1 \dots d_i}^{a_1 \dots a_i b_1 \dots b_i} R_{a_1 b_1}{}^{c_1 d_1} \dots R_{a_i b_i}{}^{c_i d_i}, \quad (4.7)$$

and $G_{ab}^{(i)}$ is the generalized ‘‘Einstein tensor’’ for the i th Lovelock Lagrangian in the hypersurface, i.e.,

$$G_{ab}^{(i)a} = -\frac{1}{2^{i+1}} \delta_{bc_1 \dots c_i d_1 \dots d_i}^{aa_1 \dots a_i b_1 \dots b_i} R_{a_1 b_1}{}^{c_1 d_1} \dots R_{a_i b_i}{}^{c_i d_i}. \quad (4.8)$$

In Eq.(4.8), the tensor $E_{acbd}^{(i)}$ has a symmetry of Riemann tensor and is defined as

$$E_{acbd}^{(i)} = \frac{1}{2} \frac{\partial L_{(i)}}{\partial R^{[ac][bd]}}. \quad (4.9)$$

Based on the above discussion, we find the left-hand side of Eq.(4.3) can be transformed into a form

$$\mathcal{G}_{ab} = \sum_{i=0}^{[n/2]} \alpha_{(i)} \left[G_{ab}^{(i)} - \frac{1}{2} \lambda^{-1} L_{(i)} K_a K_b + E_{acbd}^{(i)} (2\lambda^{-1} D^c D^d \lambda - \lambda^{-2} D^c \lambda D^d \lambda) \right]. \quad (4.10)$$

This suggests that the Hamiltonian constraint in this theory can be written as

$$\rho = \frac{1}{2} \sum_{i=0}^{[n/2]} \alpha_{(i)} L_{(i)}. \quad (4.11)$$

Similarly, the evolution equations are given by

$$p h^{ab} = \sum_{i=0}^{[n/2]} \alpha_{(i)} \left[G^{(i)ab} + E^{(i)acbd} (2\lambda^{-1} D_c D_d \lambda - \lambda^{-2} D_c \lambda D_d \lambda) \right]. \quad (4.12)$$

With the preparation in the above, we can discuss the entropy principle in this theory now. The variation of the total entropy inside the region C is still described by Eq.(3.15). We will prove the Theorem 1 in the Lovelock gravity theory order by order. We define

$$\rho_i = \frac{1}{2} L_{(i)}, \quad (4.13)$$

and

$$p_i h^{ab} = G^{(i)ab} + E^{(i)acbd} (2\lambda^{-1} D_c D_d \lambda - \lambda^{-2} D_c \lambda D_d \lambda). \quad (4.14)$$

It should be noted here: ρ_i and p_i are not the real energy density and pressure of the fluid (the energy density and pressure of the fluid are denoted by ρ and p respectively). Actually, we have

$$\rho = \sum_i \alpha_{(i)} \rho_i, \quad p = \sum_i \alpha_{(i)} p_i. \quad (4.15)$$

Consequently, the variation of the total entropy inside C can be expressed as

$$\delta S = \sum_i \alpha_{(i)} \delta S_i, \quad (4.16)$$

where δS_i is defined as

$$\delta S_i = \int_C \left[\bar{\epsilon} \frac{p_i}{2T} h^{ab} \delta h_{ab} + \frac{1}{T} \delta(\bar{\epsilon} \rho_i) \right]. \quad (4.17)$$

The second term in the integrand of the right-hand side of Eq.(4.17) can be expanded as

$$\int_C \frac{1}{T} \delta(\bar{\epsilon} \rho_i) = \frac{1}{2} \int_C \frac{1}{T} \delta(\bar{\epsilon} L_i) = \frac{1}{2} \int_C \bar{\epsilon} \frac{1}{T} \left(-G^{(i)ab} \delta h_{ab} + \mathcal{B}_i \right), \quad (4.18)$$

where \mathcal{B}_i is the total derivative term which come from the variation of L_i on the hypersurface. Explicitly, this term can be written as

$$\mathcal{B}_i = 2D_c \left(\frac{\partial L_i}{\partial R_{acbd}} D_b \delta h_{ad} \right) - 2D_b \left(D_c \frac{\partial L_i}{\partial R_{acbd}} \delta h_{ad} \right). \quad (4.19)$$

By the same token in the Einstein-Gauss-Bonnet gravity, we can rewrite the last term of Eq.(4.18) as follows

$$\begin{aligned} \frac{1}{2} \int_C \bar{\epsilon} \frac{\mathcal{B}_i}{T} &= \int_C \bar{\epsilon} \frac{1}{T} \left[D_c \left(\frac{\partial L_i}{\partial R_{acbd}} D_b \delta h_{ad} \right) - D_b \left(D_c \frac{\partial L_i}{\partial R_{acbd}} \delta h_{ad} \right) \right] \\ &= - \int_C \bar{\epsilon} \left[D_c D_d \left(\frac{1}{T} \right) \frac{\partial L_i}{\partial R_{acbd}} \delta h_{ab} + D_c \left(\frac{1}{T} \right) D_d \frac{\partial L_i}{\partial R_{acbd}} \delta h_{ab} \right] \\ &= - \int_C \bar{\epsilon} D_c D_d \left(\frac{1}{T} \right) E^{(i)acbd} \delta h_{ab}, \end{aligned} \quad (4.20)$$

where we have used an identity $D_d(\partial L_i / \partial R_{acbd}) = 0$ which can be deduced from Bianchi identity of Riemann tensor. Actually, it is easy to find

$$\begin{aligned} D_d \frac{\partial L_i}{\partial R_{acbd}} &= D_d \left(\frac{\partial L_i}{\partial R_{ac}{}^{ef}} h^{eb} h^{fd} \right) \\ &= h^{eb} D^f \left(\frac{i}{2^i} \delta_{ef e_1 \dots e_{i-1} f_1 \dots f_{i-1}}^{aca_1 \dots a_{i-1} c_1 \dots c_{i-1}} R_{a_1 c_1}{}^{e_1 f_1} \dots R_{a_{i-1} c_{i-1}}{}^{e_{i-1} f_{i-1}} \right) \\ &= \frac{i}{2^i} h^{eb} \delta_{ef e_1 \dots e_{i-1} f_1 \dots f_{i-1}}^{aca_1 \dots a_{i-1} c_1 \dots c_{i-1}} D^{[f} \left(R_{a_1 c_1}{}^{e_1 f_1} \dots R_{a_{i-1} c_{i-1}}{}^{e_{i-1} f_{i-1}} \right] \\ &= 0. \end{aligned} \quad (4.21)$$

Considering the Tolman's law, and Eqs.(4.14), (4.18), and (4.20), we find $\delta S_i = 0$ holds for each i . The total entropy thus takes an extremum value, i.e.,

$$\delta S = \sum_{i=0}^{[n/2]} \alpha_i \delta S_i = 0. \quad (4.22)$$

By now, we complete our proof of the Theorem 1 in Lovelock gravity.

In the above procedure, we can also reverse the proof by assuming that total entropy already achieves extremum, and then the evolution equations can be deduced from the Hamiltonian constraint. This can be easily seen from Eq.(4.17). Thus we arrive at a converse theorem

Theorem 2 - Consider a perfect fluid in a static n -dimensional spacetime (M, g_{ab}) in Lovelock gravity and Σ is an $(n-1)$ -dimensional hypersurface orthogonal to the static Killing vector field. Let C be a region on Σ with a boundary ∂C . Assume that the temperature of the fluid obeys Tolman's law and both Hamiltonian constraint equation and fluid equations are satisfied in C . Then the evolution equations are implied by the extrema of the total fluid entropy for a fixed total particle number in C and for all variations in which h_{ab} and its first derivatives are fixed on ∂C .

V. SPACETIME WITH AN $(n-2)$ -DIMENSIONAL MAXIMALLY SYMMETRIC SPACE

With the assumption of maximally symmetric $(n-2)$ -dimensional space, our previous work [21] has shown that the generalized Tolman-Oppenheimer-Volkoff (TOV) equation can be deduced from the (maximum) entropy principle together with the Hamiltonian constraint in Lovelock gravity theory.

According to the present discussion, the (maximum) entropy principle of perfect fluid can be realized by using of EOM in such spacetime manifestly. The n -dimensional spacetime metric is assumed as

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Psi(r)} dr^2 + r^2 \gamma_{ij} dz^i dz^j, \quad (5.1)$$

where $\gamma_{ij} dz^i dz^j$ is the metric of an $(n-2)$ -dimensional maximally symmetric space. The nontrivial components of the Riemann tensor of such spacetime are given by

$$\begin{aligned} R^{tr}_{tr} &= -2e^{-2\Psi}(\Phi'^2 - \Phi'\Psi' + \Phi''), & R^{ti}_{tj} &= \frac{e^{-2\Psi}}{r} \Phi' \delta^i_j, \\ R^{ij}_{kl} &= \frac{k - e^{-2\Psi}}{r^2} \delta^i_k \delta^j_l, & R^{ri}_{rj} &= -\frac{e^{-2\Psi}}{r} \Psi' \delta^i_j. \end{aligned} \quad (5.2)$$

where $k = 0, \pm 1$ corresponds to the sectional curvature of the maximally symmetric space and the prime denotes the derivative with respect to radial coordinate r . The gravitational equations with perfect fluid can be put in the form

$$\kappa_n^2 \rho = \frac{1}{r^{n-2}} \frac{d}{dr} \left\{ \sum_{i=0}^{[n/2]} \frac{\alpha_i (n-2)!}{2(n-2i-1)!} r^{n-1-2i} (k - e^{-2\Psi})^i \right\}, \quad (5.3)$$

which comes from $\mathcal{G}_t^t = \kappa_n^2 T_t^t$, and

$$\begin{aligned} \kappa_n^2 p &= \sum_{i=0}^{[n/2]} \frac{i \alpha_i (n-2)!}{(n-2i-1)!} \frac{e^{-2\Psi} \Phi'}{r} \left(\frac{k - e^{-2\Psi}}{r^2} \right)^{i-1} \\ &\quad - \sum_{i=0}^{[n/2]} \frac{\alpha_i (n-2)!}{2(n-2i-2)!} \left(\frac{k - e^{-2\Psi}}{r^2} \right)^i, \end{aligned} \quad (5.4)$$

which is given by $\mathcal{G}_r^r = \kappa_n^2 T_r^r$.

The total entropy and total particle number inside ∂C can be written as

$$S = \omega_k \int_0^R s e^\Psi r^{n-2} dr, \quad (5.5)$$

$$N = \omega_k \int_0^R n e^\Psi r^{n-2} dr, \quad (5.6)$$

where $\omega_k := \int d^{n-2} z \sqrt{\gamma}$ is the volume of the maximally symmetric space with the sectional curvature k , and R is the radius of spacial boundary ∂C .

With additional assumptions of the Tolman's law (note that $\sqrt{-\lambda} = e^\Phi$) and fixed particle number N inside ∂C , we can get that $\Phi' = -T'/T$ and the variation of total entropy

$$\delta S = \omega_k \int_0^R \left(\frac{p}{T} e^\Psi \delta \Psi + \frac{1}{T} \delta(e^\Psi \rho) \right) r^{n-2} dr. \quad (5.7)$$

Substituting the gravitational equations Eqs.(5.3) and (5.4) into the above equation and performing the integration by parts, we finally get $\delta S = 0$ when the boundary condition of Ψ : $\delta \Psi(R) = 0$ is imposed.

Thus we realize the entropy principle in case of spacetime with $(n-2)$ -dimensional maximally symmetric space. We used the boundary condition $\delta \Psi(R) = 0$ in our derivation. For the metric of Eq.(5.1), the Misner-Sharp energy takes the form [6, 25]

$$m(r) = \frac{\omega_k}{2\kappa_n^2} \sum_{i=0}^{[n/2]} \frac{\alpha_i (n-2)!}{(n-1-2i)!} r^{n-1-2i} (k - e^{-2\Psi})^i. \quad (5.8)$$

Clearly, the boundary condition here has its physical meaning of fixed Misner-Sharp energy inside the spacial boundary. The volume of spacial boundary $A(R) = \int R^{n-2} \sqrt{\gamma} d^{n-2} z$ is held fixed as well as the total particle number N inside ∂C . So $\delta m(R) = 0$, $\delta A(R) = 0$, and $\delta N = 0$ define an isolated system quasilocally.

In addition, we can reverse the procedure to get the so-called generalized TOV equation by assuming that both the Hamiltonian constraint {here is the tt component of gravitational equations Eq.(5.3)} and equilibrium state of perfect fluid have been already satisfied. Starting from Eq.(5.7) with $\delta S = 0$ and the exact expression of ρ , i.e., Eq.(5.3), we obtain the evolution equation as

$$\begin{aligned} \kappa_n^2 p = & -\frac{T'}{T} \sum_{i=0}^{[n/2]} \frac{i\alpha_i(n-2)!}{(n-2i-1)!} \frac{e^{-2\Psi}}{r} \left(\frac{k - e^{-2\Psi}}{r^2} \right)^{i-1} \\ & - \sum_{i=0}^{[n/2]} \frac{\alpha_i(n-2)!}{2(n-2i-2)!} \left(\frac{k - e^{-2\Psi}}{r^2} \right)^i. \end{aligned} \quad (5.9)$$

Note that the thermodynamic first law in terms of densities Eq.(3.5) and Gibbs-Duham relation Eq.(3.6) tells us that

$$dp = sdT + nd\mu, \quad (5.10)$$

together with Eq.(3.10) we can get the following relation

$$\frac{T'}{T} = \frac{p'}{\rho + p}. \quad (5.11)$$

So the evolution equation can be written as a relation among energy density, pressure, and metric components. This is the so-called generalized TOV equation in Lovelock gravity

$$\begin{aligned} \kappa_n^2 p = & -\frac{p'}{\rho + p} \sum_{i=0}^{[n/2]} \frac{i\alpha_i(n-2)!}{(n-2i-1)!} \frac{e^{-2\Psi}}{r} \left(\frac{k - e^{-2\Psi}}{r^2} \right)^{i-1} \\ & - \sum_{i=0}^{[n/2]} \frac{\alpha_i(n-2)!}{2(n-2i-2)!} \left(\frac{k - e^{-2\Psi}}{r^2} \right)^i. \end{aligned} \quad (5.12)$$

More compactly, the generalized TOV equation can be written as

$$\frac{\partial m}{\partial \Psi} \frac{p'}{\rho + p} = - \left[p\omega_k r^{n-2} + m' - \frac{\partial m}{\partial \Psi} \Psi' \right], \quad (5.13)$$

where $m(\Psi)$ is the Misner-Sharp energy Eq.(5.8) and here it is understood as a function of Ψ .

As a little worm-up, we can see that at least in the case of spacetime with maximally symmetric space, there do exist the equivalent description between the geometrical equations and the laws of thermodynamics without any man-made input since the the boundary condition can be finally realized as the necessary condition for an isolated system quasilocally.

The next section will focus on the static case, and talk about the boundary conditions and its physical meaning.

VI. THE BOUNDARY CONDITIONS AND QUASILOCALLY ISOLATED SYSTEM

In the previous sections, we have proved that the total entropy inside the spacelike $(n-2)$ -dimensional surface ∂C will take extremal value once the Tolman's law and both gravitational and fluid equations are held. To get this conclusion, we have assumed that the total particle number N inside the spacial region C is held fixed and the variations δh_{ab} and $D_a \delta h_{bc}$ are both vanishing on ∂C . The meaning of the condition δN is straightforward-there is no effective matter communication between the inside and outside of ∂C . But the physical meaning of the conditions $\delta h_{ab} = D_a \delta h_{bc} = 0$ is still unclear so far.

For a thermodynamic system by usual matter, the total entropy will take maximal value for equilibrium state when the system is isolated. In this case, the total energy E , volume V , and particle number N of the system are all held fixed, i.e., we have

$$\delta E = 0, \quad \delta V = 0, \quad \delta N = 0. \quad (6.1)$$

However, when gravity is taken into account, that is for a self-gravitating one, the phrase “isolated system” becomes ambiguous since the gravitational interaction is a long range force. In Einstein gravity theory, the entropy extremum of relativistic self-bound fluid in stationary axisymmetric spacetime has been studied by Katz and Manor [26], they have presented the condition for global isolation with globally defined quantities such as total mass energy and total angular momentum. For a quasilocal system, the (maximum) entropy problem of self-gravitating matter has been studied by many authors [11, 12], together with our previous paper [21], where all the discussions are based on the assumption of spacetime with maximally symmetric space. Furthermore the quasilocal isolation is realized by requiring that the Misner-Sharp energy $m(R)$ inside a fixed radius R does not change under the variation. The condition of fixed particle number $N(R)$ appears as a Lagrange multiplier. So we see that at

least in spherically symmetric spacetime, the quasilocal realization of isolation requires fixed Misner-Sharp energy $m(R)$, total particle number $N(R)$ and fixed spacial area $\omega_k R^{n-2}$.

In this section, we will give the definition of isolated system quasilocally for a more general static spacetime in Lovelock gravity theory which has implied by the boundary conditions mentioned above that was used to proof the extremum of total entropy.

First of all, a quasilocal system should have finite spacial volume or more rigorously, such system has spacial boundary. We will denote the product of surface ∂C with segments of timelike Killing vector field K^a 's integral curve as a timelike boundary of spacetime manifold M as $^{(n-1)}B$ with a unit norm n^a . The induced metric and extrinsic curvature tensor of this timelike boundary $^{(n-1)}B$ denoted as γ_{ab} and Θ_{ab} have the following form

$$\gamma_{ab} = g_{ab} - n_a n_b, \quad \Theta_{ab} = \gamma_a^c \gamma_b^d \nabla_c n_d. \quad (6.2)$$

Once a timelike boundary is imposed, the well-defined variational principle requires a boundary term to cancel the total derivatives that produce surface integrals involving the derivative of δg_{ab} normal to the boundary $^{(n-1)}B$. For The Lovelock gravity theory, the boundary term can be written as [27, 28]

$$I_b = - \int \tilde{\epsilon} \sum_{i=0}^{[n/2]} \sum_{s=0}^{i-1} \frac{(-)^{i-s} i \alpha_i}{2^s (2i - 2s - 1)} \mathcal{H}^{(i)}, \quad (6.3)$$

where $\tilde{\epsilon}$ is the volume element of $^{(n-1)}B$ associated with the induced metric γ_{ab} and $\mathcal{H}^{(i)}$ is defined as

$$\mathcal{H}^{(i)} = \delta_{b_1 \dots b_{2i-1}}^{a_1 \dots a_{2i-1}} \mathcal{R}^{b_1 b_2}_{a_1 a_2} \dots \mathcal{R}^{b_{2s-1} b_{2s}}_{a_{2s-1} a_{2s}} \Theta_{a_{2s+1}}^{b_{2s+1}} \dots \Theta_{a_{2i-1}}^{b_{2i-1}}. \quad (6.4)$$

At this point, together with the boundary term, we have the total action for a quasilocal gravitational system in Lovelock theory. It is now straightforward to show that one can use the generalized Hamilton-Jacobi method [5] to construct a divergence free quasilocal stress-energy tensor defined on the timelike boundary $^{(n-1)}B$ as [27, 28]

$$T_b^a = - \sum_{i=0}^{[n/2]} \sum_{s=0}^{i-1} \frac{(-)^{i-s} i \alpha_i}{2^s (2i - 2s - 1)} \mathcal{H}_b^{(i,s)a}, \quad (6.5)$$

where $\mathcal{H}_b^{(i,s)a}$ is

$$\mathcal{H}_b^{(i,s)a} = \delta_{b_1 \dots b_{2i-1} b}^{a_1 \dots a_{2i-1} a} \tilde{R}^{b_1 b_2}_{a_1 a_2} \dots \tilde{R}^{b_{2s-1} b_{2s}}_{a_{2s-1} a_{2s}} \Theta_{a_{2s+1}}^{b_{2s+1}} \dots \Theta_{a_{2i-1}}^{b_{2i-1}}, \quad (6.6)$$

and \tilde{R}_{abcd} is the intrinsic curvature tensor of the timelike boundary $^{(n-1)}B$, γ_{ab} .

Since the spacetime is static, the extrinsic curvature of spacelike hypersurface is vanishing. So we can decompose the intrinsic and extrinsic curvature tensor of $^{(n-1)}B$ along the time slice as follows

$$\tilde{R}_{abcd} = \hat{R}_{abcd} - 4u_{[a} \frac{\hat{D}_{b]} \hat{D}_{[c} \sqrt{-\lambda}}{\sqrt{-\lambda}} u_{d]}, \quad (6.7)$$

$$\Theta_b^a = k_b^a + u^a u_b n_c a^c, \quad (6.8)$$

where \hat{R}_{abcd} is the intrinsic curvature tensor of $(n-2)$ -dimensional surface $(\partial C, \sigma_{ab})$ with the covariant derivative operator denoted as \hat{D}_a which can be viewed as a submanifold embedded in spacelike hypersurface Σ with a unit norm n^a . $\sigma_{ab} = h_{ab} - n_a n_b$ and $k_{ab} = -\sigma_a^c D_c n_b$ are induced metric and extrinsic curvature tensor of ∂C 's, and a^b is the acceleration vector of the static observer.

With the form of stress-energy tensor defined in Eq.(6.5), one can obtain the energy density observed by the static observer as

$$\varepsilon = T_b^a u_a u^b = - \sum_{i=0}^{[n/2]} \sum_{s=0}^{i-1} \frac{(-)^{i-s} i \alpha_i}{2^s (2i - 2s - 1)} t^{(i,s)}, \quad (6.9)$$

where

$$t^{(i,s)} = \delta_{b_1 \dots b_{2i-1} b}^{a_1 \dots a_{2i-1} a} u_a u^b \hat{R}^{b_1 b_2}_{a_1 a_2} \dots \hat{R}^{b_{2s-1} b_{2s}}_{a_{2s-1} a_{2s}} k_{a_{2s+1}}^{b_{2s+1}} \dots k_{a_{2i-1}}^{b_{2i-1}}, \quad (6.10)$$

which is the total energy density of the region C quasilocally defined on its spacial boundary ∂C . So the total energy of the region C as a thermodynamic quantity can be written as the following integration

$$E = \int \hat{\epsilon} \varepsilon, \quad (6.11)$$

where $\hat{\varepsilon}$ is the volume element of ∂C .

Now let us study the physical implication of boundary conditions we have imposed on variation of spacelike hypersurface Σ 's induced metric h_{ab} and its first derivatives. First, we noted that the static observer's four velocity u^a is hypersurface orthogonal to Σ . This means that the (1,1)-type tensor field h_b^a is a projection operator which equals to a Kronecker delta symbol of Σ when restricted on the hypersurface Σ . Then we can conclude that δh^{ab} also vanishes on the spatial boundary ∂C because we can use the relation $h^{ab}h_{bc} = h_c^a$ to deduce that

$$\delta h^{ab} = h^{bc}\delta h_c^a - h^{ac}h^{bd}\delta h_{cd} = -h^{ac}h^{bd}\delta h_{cd}, \quad (6.12)$$

where the variation of h_b^a vanishes since the variation is restricted on Σ . Second, we find the following variational relation by using the fact that n^a is the unit norm of the surface ∂C embedded in spacelike hypersurface Σ

$$\delta n_a = \frac{1}{2}n_a n^b n^c \delta h_{bc}. \quad (6.13)$$

The (1-1)-type tensor σ_b^a can also be viewed as a projection operator of the surface ∂C and equals the Kronecker delta symbol when restricted on it. Thus, the above relation tells us that the variation of induced metric σ_{ab} of ∂C together with its all possible index form will vanish when restricted on the surface ∂C .

Based on the above discussion, the variation of quasilocal energy Eq.(6.11) yields the following form

$$\begin{aligned} \delta E &= \int \delta(\hat{\varepsilon})\varepsilon + \int \hat{\varepsilon}\delta(\varepsilon) \\ &= \int \frac{1}{2}\hat{\varepsilon}\varepsilon\sigma^{ab}\delta\sigma_{ab} \\ &\quad - \int \hat{\varepsilon} \sum_{i=0}^{[n/2]} \sum_{s=0}^{i-1} \frac{(-)^{i-s}i\alpha_i}{2^s(2i-2s-1)} \left[\delta_{b_1 \dots b_{2i-1}b}^{a_1 \dots a_{2i-1}a} \delta(u_a u^b) \hat{R}^{b_1 b_2}_{a_1 a_2} \dots \hat{R}^{b_{2s-1} b_{2s}}_{a_{2s-1} a_{2s}} k_{a_{2s-1}}^{b_{2s-1}} \dots k_{a_{2i-1}}^{b_{2i-1}} \right. \\ &\quad \left. + s \delta_{b_1 \dots b_{2i-1}b}^{a_1 \dots a_{2i-1}a} u_a u^b \hat{R}^{b_1 b_2}_{a_1 a_2} \dots \delta \hat{R}^{b_{2s-1} b_{2s}}_{a_{2s-1} a_{2s}} k_{a_{2s-1}}^{b_{2s-1}} \dots k_{a_{2i-1}}^{b_{2i-1}} \right. \\ &\quad \left. + (2i-2s-1) \delta_{b_1 \dots b_{2i-1}b}^{a_1 \dots a_{2i-1}a} u_a u^b \hat{R}^{b_1 b_2}_{a_1 a_2} \dots \hat{R}^{b_{2s-1} b_{2s}}_{a_{2s-1} a_{2s}} k_{a_{2s-1}}^{b_{2s-1}} \dots \delta k_{a_{2i-1}}^{b_{2i-1}} \right], \end{aligned} \quad (6.14)$$

where

$$\delta(u_a u^b) = \delta h_a^b, \quad (6.15)$$

$$\delta \hat{R}^{ab}_{cd} = \sigma^{ae} \sigma^{bf} \sigma_{dg} \hat{D}_f (\sigma^{gm} \hat{D}_e \delta \sigma_{cm} + \sigma^{gm} \hat{D}_c \delta \sigma_{em} - \sigma^{gm} \hat{D}_m \delta \sigma_{ec}) + 2 \hat{R}^b_{cd} \delta \sigma^{ae} + \hat{R}^{ab}_c{}^g \delta \sigma_{dg}, \quad (6.16)$$

$$\delta k_b^a = -\frac{1}{2} \sigma_b^c h^{ae} (D_c \delta h_{de} + D_d \delta h_{ce} - D_e \delta h_{cd}) - D_c n^a \delta \sigma_b^c - \sigma_b^c D_c \delta n^a. \quad (6.17)$$

Note the \hat{D}_a is the covariant derivative operator which is compatible with σ_{ab} and ∂C has no boundary. All the variational terms in δE can be finally changed into the form which contains δh_{ab} and $D_a \delta h_{bc}$. After considering the integration on ∂C , the boundary conditions stated in the theorems finally yield

$$\delta E = 0, \quad (6.18)$$

which is nothing but the physical requirement that no energy exchange with environment of an isolated system.

Since the total energy of the region C is defined quasilocally, a natural choice of volume of such a system is now the surface area of the region C , that is the volume of the $(n-2)$ -dimensional surface ∂C

$$A = \int \hat{\varepsilon}. \quad (6.19)$$

According to our boundary conditions, it is easy to see that the variation of this volume vanishes, i.e., $\delta A = 0$.

Comparing our results with the thermodynamic isolated system by the usual matter, we can now claim that the boundary conditions together with the fixed total particle number N stated in theorems imply an isolated system quasilocally in Lovelock gravity theory with

$$\delta E = 0, \quad \delta A = 0, \quad \delta N = 0. \quad (6.20)$$

As we have seen, the two boundary conditions $\delta h_{ab} = 0$ and $D_a \delta h_{bc} = 0$ on the spacelike $(n-2)$ -dimensional surface ∂C are necessary conditions to define a quasilocally isolated system in Lovelock gravity theory.

If we relax one of the boundary conditions, then we can expect to get the variational relation among the total entropy and other thermodynamic quantities. Let us take Einstein gravity as a simple example. We will relax the boundary condition $D_a \delta h_{bc} = 0$ on ∂C and see the result of entropy variation.

Without the assumption of vanishing of the first derivative of all variations of induced metric on ∂C , the variation of total entropy now is nonzero even if the equations of motion of gravity are satisfied.

$$\begin{aligned}
\delta S &= \int_C \left[\bar{\epsilon} \frac{p}{2T} h^{ab} \delta h_{ab} + \frac{1}{T} \delta(\bar{\epsilon} \rho) \right] \\
&= \int_C \left[\bar{\epsilon} \frac{p}{2T} h^{ab} \delta h_{ab} + \frac{1}{2T} \delta(\bar{\epsilon} R) \right] \\
&= \int_C \bar{\epsilon} \left[h^{a[b} h^{d]c} D_c D_d \left(\frac{1}{T} \right) \delta h_{ab} + \frac{1}{2T} D^a D^b \delta h_{ab} - \frac{1}{2T} D^a (h^{bc} D_a \delta h_{bc}) \right] \\
&= \int_C \bar{\epsilon} \left[D^a \left(\frac{1}{2T} D^b \delta h_{ab} \right) - D^a \left(\frac{1}{2T} h^{bc} D_a \delta h_{bc} \right) \right] \\
&= \int_{\partial C} \frac{\hat{\epsilon}}{T} \left[\frac{1}{2} n^a D^b \delta h_{ab} - \frac{1}{2} n^a D_a (h^{bc} \delta h_{bc}) \right] \\
&= -\frac{1}{2} \int_{\partial C} \frac{\hat{\epsilon}}{T} n^a D_a (h^{bc} \delta h_{bc}), \tag{6.21}
\end{aligned}$$

In the last step, we have used the fact that $D^b \delta h_{ab} = \sigma^{be} D_e \delta h_{bd} = 0$ because $\delta h_{ab} = 0$ on ∂C .

On the other hand, one can define the quasilocal energy inside ∂C according to Eqs.(6.9), (6.10), and (6.11) when limited in the Einstein gravity case as

$$E = - \int_{\partial C} \hat{\epsilon} \sigma^a_b D_a n^b. \tag{6.22}$$

If we vary the above energy only with $\delta h_{ab} = 0$ on ∂C , then we will find

$$\begin{aligned}
\delta E &= - \int_{\partial C} \hat{\epsilon} h^a_b \delta C^b_{ac} n^c = -\frac{1}{2} \int_{\partial C} \hat{\epsilon} h^a_b h^{bd} (D_a \delta h_{cd} + D_c \delta h_{ad} - D_d \delta h_{ac}) n^c \\
&= -\frac{1}{2} \int_{\partial C} \hat{\epsilon} n^a D_a (h^{bc} \delta h_{bc}). \tag{6.23}
\end{aligned}$$

Thus, when ∂C is an isothermal boundary, the variation of total entropy inside ∂C can be written as

$$\delta S = \frac{1}{T} \delta E. \tag{6.24}$$

This is nothing but the thermodynamic first law of the isometrical system since we have fixed the induced metric on ∂C .

VII. CONCLUSIONS AND DISCUSSION

In this paper, we have shown that the (maximum) entropy principle of the perfect fluid in curved spacetimes can be realized by using the gravitational equations in the Lovelock gravity theory. This result has been put into the Theorem 1. Comparing to our previous paper, the symmetry of an $(n-2)$ -dimensional maximally symmetric space has not been imposed, and the only symmetry required here is the static condition.

For the traditional thermodynamics in flat spacetime, the entropy of matter must take maximal value in an equilibrium state if the system is isolated. When backreaction is encountered, that is for self-gravitating system, it seems that the requirement for isolation at least includes following conditions: First, the system inside an $(n-2)$ -dimensional spacelike surface ∂C should have a fixed total particle number. Second, the induced metric h_{ab} on Σ and its first derivatives should be fixed on ∂C . Physically, the first condition implies that the system has no effective particle communication with the outside region. The second one implies two physical explanations, one is the volume of the system which is quasilocally defined as the spacial region C 's surface area should keep fixed, the other is the total quasilocal energy of the system does not change under the variations of the matter fields. Thus these two conditions in Theorem 1 and 2 will give the definition of an isolated system quasilocally when backreaction is taken into account.

We have just shown that the total entropy of the perfect fluid for this isolated system must take extremum value when both the gravitational and fluid equations are satisfied. However, we do not know the extremum is a maximum or a minimum at present. To confirm the state is a real equilibrium state, one has to perform a second order variation and analyze the stability conditions of the system. The maximum entropy principle in general relativity with stability analysis in spherical symmetric system has

been studied by Roupas [29]. This is an interesting point and needs further study in static spacetime. This is also the reason that we have put the “maximum” inside brackets in the title and the main part of this paper.

It is still unclear whether this (maximum) entropy principle can be extended to other gravity theories or not. We believe this deep connection between gravity theory and thermodynamics is still there and waiting for people to uncover.

Finally, is it possible to apply the (maximum) entropy principle to a stationary spacetime? This is also unclear up to date and deserves to be studied carefully in the future.

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